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Series solutions for a nonlinear model of combined convective and radiative cooling of a spherical body

Shijun Liao [a,*], Jian Su [b], Allen T. Chwang [c]

[a] *School of Naval Architecture, Ocean and Civil Engineering, Shanghai Jiao Tong University, Shanghai, China*
[b] *Nuclear Engineering Program COPPE, Universidade Federal do Rio de Janeiro, CP 68509, Rio de Janeiro 21945-970, Brazil*
[c] *Department of Mechanical Engineering, University of Hong Kong, Hong Kong*

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Abstract

An analytic approach based on the homotopy analysis method is proposed to solve a nonlinear model of combined convective and radiative cooling of a spherical body. An explicit series solution is given, which agrees well with the exact or numerical solutions. Our series solutions indicate that, for the nonlinear model of combined convective and radiative cooling of a spherical body, the temperature on the surface of the body decays more quickly for larger values of the Biot number Bi and/or the radiation–conduction parameter N_{rc}. Different from traditional analytic techniques based on eigenfunctions and eigenvalues for linear problems, our approach is independent of the concepts of eigenfunctions and eigenvalues, and besides is valid for nonlinear problems in general. This analytic method provides us with a new way to obtain series solutions of unsteady nonlinear heat conduction problems, which are valid for all dimensionless times $0 \leqslant \tau < +\infty$.

Keywords: Combined convective and radiative cooling; Unsteady; Series solution

1. Introduction

Transient heat conduction in a solid with combined convective and radiative cooling on the surfaces has received relatively little attention in the literature despite its relevance in various technological applications such as aerothermodynamic heating of spaceships and satellites, nuclear reactor thermohydraulics and glass manufacturing [1–3].

Haji-Sheikh and Sparrow [4] used the Monte Carlo technique to obtain solutions for a plate subjected to simultaneous boundary convection and radiation. Crosbie and Viskanta [5] analysed transient cooling and heating of a plate by combined convection and radiation. The transient heat conduction equation and the boundary conditions are transformed into a nonlinear Volterra integral equation of the second kind for the surface temperature. Davies [6] applied the heat balance integral technique to obtain an approximate solution for the general conditions of a plate in a non-zero temperature environment. Sundén [7] presented numerical solutions based on the finite difference method of the thermal response of a composite slab subjected to a time-varying incident heat flux on one side and combined convective and radiative cooling on the other side. Sundén [8] applied the same technique to assess the thermal response of a circular cylindrical shell due to a time-varying incident surface heat flux while cooled by combined convection and radiation. Parang et al. [2] solved the problem of inward solidification of a liquid in cylindrical and spherical geometries due to combined convective and radiative cooling by the regular perturbation method.

The special limiting case of transient radiative cooling of a spherical body has been analyzed recently by using improved lumped-differential formulations obtained

* Corresponding author. Tel.: +86 21 6293 2676; fax: +86 21 6293 3156.
E-mail addresses: sjliao@sjtu.edu.cn (S. Liao), sujian@con.ufrj.br (J. Su), atchwang@hkucc.hku.hk (A.T. Chwang).

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through two-point Hermite approximations for integrals [9–11]. The same approximate analytic approach has been applied in various transient heat conduction problems subjected to convective boundary conditions [12–14]. An explicit series solution is given here for the first time (to the best of our knowledge) for the combined convective and radiative cooling, which is valid for all dimensionless times.

2. Basic equations

Consider transient combined convective and radiative cooling of a spherical body of radius R, initially at a uniform temperature T_i. At $t = 0$, the spherical body is suddenly exposed to an environment of a constant fluid temperature T_f and a constant radiation sink temperature T_s. It is assumed that the spherical body is homogeneous, isotropic and opaque. The thermal conductivity k is temperature-dependent, while the density ρ and specific heat c_p are assumed to be constant.

The mathematical formulation of the problem is given by

$$\rho c_p \frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 k(T)\frac{\partial T}{\partial r}\right], \quad \text{in } r < R, \text{ for } t > 0, \tag{1a}$$

with initial and boundary conditions taken as

$$T(r,0) = T_i, \quad \text{in } r \leqslant R, \text{ at } t = 0, \tag{1b}$$

$$-k(T)\frac{\partial T}{\partial r} = h(T - T_f) + \epsilon\sigma(T^4 - T_s^4), \quad \text{at } r = R, \text{ for } t > 0, \tag{1c}$$

$$\frac{\partial T}{\partial r} = 0, \quad \text{at } r = 0, \text{ for } t > 0, \tag{1d}$$

where T denotes the temperature, t the time, r the spatial coordinate, h the convective heat transfer coefficient, ϵ the surface emissivity, and σ the Stefan–Boltzmann constant, respectively.

It is convenient to introduce the adiabatic surface temperature T_a, defined by

$$h(T_a - T_f) + \epsilon\sigma(T_a^4 - T_s^4) = 0, \tag{2}$$

which gives

$$hT_f + \epsilon\sigma T_s^4 = hT_a + \epsilon\sigma T_a^4.$$

Substituting it into (1c), the corresponding boundary condition can be rewritten as

$$-k(T)\frac{\partial T}{\partial r} = h(T - T_a) + \epsilon\sigma(T^4 - T_a^4), \quad \text{at } r = R, \text{ for } t > 0. \tag{3}$$

The above mathematical formulations can now be rewritten in dimensionless form as

$$\frac{\partial \theta}{\partial \tau} = \frac{1}{\eta^2}\frac{\partial}{\partial \eta}\left[\lambda(\theta)\eta^2\frac{\partial \theta}{\partial \eta}\right], \quad \text{in } \eta < 1, \text{ for } \tau > 0, \tag{4a}$$

subject to the initial and boundary conditions

$$\theta(\eta, 0) = 1, \quad \text{in } \eta \leqslant 1, \text{ at } \tau = 0, \tag{4b}$$

$$-\lambda(\theta)\frac{\partial \theta}{\partial \eta} = Bi(\theta - \theta_a) + N_{rc}(\theta^4 - \theta_a^4), \quad \text{at } \eta = 1, \text{ for } \tau > 0, \tag{4c}$$

$$\frac{\partial \theta}{\partial \eta} = 0, \quad \text{at } \eta = 0, \text{ for } \tau > 0, \tag{4d}$$

where the dimensionless parameters are defined by

$$\theta = \frac{T}{T_i}, \quad \eta = \frac{r}{R}, \quad \tau = \frac{\alpha_0 t}{R^2}, \quad Bi = \frac{hR}{k_0},$$

$$N_{rc} = \frac{\epsilon\sigma R T_i^3}{k_0}, \quad \alpha_0 = \frac{k_0}{\rho c_p}, \quad \lambda = \frac{k}{k_0},$$

in which k_0 is a reference thermal conductivity and α_0 is a reference thermal diffusivity.

Without loss of generality, we consider the case in which the thermal conductivity varies linearly with the temperature, given by

$$k = k_0(1 + bT). \tag{5}$$

The dimensionless thermal conductivity can be written as

$$\lambda(\theta) = 1 + \beta\theta, \tag{6}$$

where $\beta = bT_i/k_0$. The mathematical model for this case is as follows:

$$\frac{\partial \theta}{\partial \tau} = \frac{1}{\eta^2}\frac{\partial}{\partial \eta}\left[(1 + \beta\theta)\eta^2\frac{\partial \theta}{\partial \eta}\right], \quad \text{in } \eta < 1, \text{ for } \tau > 0, \tag{7a}$$

subject to the initial and boundary conditions:

$$\theta(\eta, 0) = 1, \quad \text{in } \eta \leqslant 1, \text{ at } \tau = 0, \tag{7b}$$

$$-(1 + \beta\theta)\frac{\partial \theta}{\partial \eta} = Bi(\theta - \theta_a) + N_{rc}(\theta^4 - \theta_a^4), \quad \text{at } \eta = 1, \text{ for } \tau > 0, \tag{7c}$$

$$\frac{\partial \theta}{\partial \eta} = 0, \quad \text{at } \eta = 0, \text{ for } \tau > 0. \tag{7d}$$

It can be seen that the problem is governed by four dimensionless parameters, θ_a, β, Bi and N_{rc}. The radiation–conduction parameter, N_{rc}, is conceptually analog to the Biot number, Bi, which is the governing parameter for convective cooling.

When $\beta = 0$ and $N_{rc} = 0$, the above equations become linear and have the exact solution expressed by a series of eigenfunctions and eigenvalues, i.e.

$$\theta(\eta, \xi) = \sum_{m=1}^{+\infty} a_m \frac{\sin(\lambda_m \eta)}{(\lambda_m \eta)} \exp(-\lambda_m^2 \xi), \tag{8}$$

where

$$a_m = \frac{2(\sin\lambda_m - \lambda_m\cos\lambda_m)}{(\lambda_m - \sin\lambda_m\cos\lambda_m)}$$

and the eigenvalue λ_m satisfies the algebraic equation

$$\cos x + (Bi - 1)\frac{\sin x}{x} = 0. \tag{9}$$

However, the concept of eigenfunctions and eigenvalues is inherently related to linear problems. When $\beta \neq 0$ and/or $N_{rc} \neq 0$, the above equations are fully nonlinear, and thus eigenfunctions and eigenvalues have no meanings at all. Perturbation techniques can be applied to the nonlinear cases by regarding τ as a small variable. However, such kind of solutions are often only valid for small time τ. To the best of our knowledge, no one has reported any analytic solution of the considered problem, which would be valid for any dimensionless times $0 \leqslant \tau < +\infty$.

Recently, a kind of analytic method, namely the homotopy analysis method (HAM) [15], was developed to solve highly nonlinear problems. Homotopy [16] is a basic concept in topology [17], and some numerical techniques such as the continuation method [18] and the homotopy continuation method [19] were developed. Different from perturbation techniques [20], the homotopy analysis method does not depend upon any small or large parameters and thus is valid for most nonlinear problems in science and engineering. Besides, it logically embraces other non-perturbation techniques such as Lyapunov's small parameter method [21], the δ-expansion method [22], and Adomian's decomposition method [23], as proved by Liao in his book [15]. The homotopy analysis method has been successfully applied to many nonlinear problems [24–37]. In this paper, an analytic approach based on the homotopy analysis method is proposed to solve the nonlinear model of combined convective and radiative cooling of a spherical body, and an explicit series solution valid for all dimensionless time is given.

3. Series solutions given by the HAM

By means of Williams and Rhyne's [38] similarity transformation

$$\xi = 1 - \exp(-\alpha\tau), \tag{10}$$

where $\alpha > 0$ is a constant to be chosen later, the original equation (7a) becomes

$$\alpha(1-\xi)\theta_\xi = \theta_{\eta\eta} + \left(\frac{2}{\eta}\right)\theta_\eta + \beta\left[\theta\theta_{\eta\eta} + \left(\frac{2}{\eta}\right)\theta\theta_\eta + \theta_\eta^2\right], \tag{11a}$$

subject to the initial condition

$$\theta = 1, \quad \text{in } \eta < 1, \text{ at } \xi = 0 \tag{11b}$$

and the boundary conditions

$$\theta_\eta = 0, \quad \text{at } \eta = 0, \text{ for } 0 < \xi \leqslant 1, \tag{11c}$$

$$-(1+\beta\theta)\theta_\eta = Bi(\theta - \theta_a) + N_{rc}(\theta^4 - \theta_a^4), \quad \text{at } \eta = 1, \text{ for } 0 < \xi \leqslant 1, \tag{11d}$$

where the subscript denotes the derivative with respect to ξ or η.

The solution $\theta(\xi,\eta)$ of Eqs. (11a)–(11d) can be expressed by a series in the form

$$\theta(\xi,\eta) = \sum_{m=1}^{+\infty} a_{m,0}\xi^m + \sum_{m=1}^{+\infty}\sum_{n=2}^{+\infty} a_{m,n}\xi^m\eta^n, \tag{12}$$

where $a_{m,n}$ is a coefficient to be determined later. The above expression automatically satisfies the boundary condition (11c). It is rather general and our approximations should obey it. This expression is so important in the frame of the homotopy analysis method that it is regarded as a rule, called the *Rule of Solution Expression* for $\theta(\xi,\eta)$.

Physically, it is obvious that $\theta = 1$ when $\xi = 0$, corresponding to $t = 0$; and $\theta = \theta_a$ when $\xi = 1$, corresponding to $t \to +\infty$. These results can be also obtained directly from Eqs. (11a)–(11d). Therefore, under the *Rule of Solution Expression* (12) and from the initial condition (11b) and the boundary condition (11c), it is natural for us to choose an initial approximation

$$\theta_0 = 1 + (\theta_a - 1)\xi + \gamma\xi(1-\xi)\eta^2, \tag{13}$$

where γ is a constant to be chosen later. Under the *Rule of Solution Expression* (12) and from the governing Eq. (11a), it is natural to choose the auxiliary linear operator

$$\mathscr{L}[\theta] = \frac{\partial^2\theta}{\partial\eta^2} + \left(\frac{2}{\eta}\right)\frac{\partial\theta}{\partial\eta}, \tag{14}$$

which satisfies

$$\mathscr{L}\left[C_1 + \frac{C_2}{\eta}\right] = 0. \tag{15}$$

From Eq. (11d), we define at $\eta = 1$ an auxiliary linear operator

$$\mathscr{L}_b[\theta] = \theta_\eta + Bi\theta, \quad \text{at } \eta = 1, \text{ for } 0 < \xi \leqslant 1. \tag{16}$$

From Eq. (11a), we define a non-linear operator

$$\begin{aligned}\mathscr{A}[\Theta(\eta,\xi;q)] &= \frac{\partial^2\Theta(\eta,\xi;q)}{\partial\eta^2} + \left(\frac{2}{\eta}\right)\frac{\partial\Theta(\eta,\xi;q)}{\partial\eta} - \alpha(1-\xi)\frac{\partial\Theta(\eta,\xi;q)}{\partial\xi} \\ &\quad + \beta\left[\Theta(\eta,\xi;q)\frac{\partial^2\Theta(\eta,\xi;q)}{\partial\eta^2} + \left(\frac{2}{\eta}\right)\Theta(\eta,\xi;q)\frac{\partial\Theta(\eta,\xi;q)}{\partial\eta}\right] \\ &\quad + \beta\left[\frac{\partial\Theta(\eta,\xi;q)}{\partial\eta}\right]^2,\end{aligned} \tag{17}$$

where $q \in [0,1]$ is an embedding parameter and $\Theta(\eta,\xi;q)$ is a kind of mapping of $\theta(\xi;\eta)$. From (11d), we define a nonlinear operator at the boundary $\eta = 1$, i.e.

$$\begin{aligned}\mathscr{B}[\Theta(\eta,\xi;q)] &= [1+\beta\Theta(\eta,\xi;q)]\frac{\partial\Theta(\eta,\xi;q)}{\partial\eta} + Bi[\Theta(\eta,\xi;q) - \theta_a] \\ &\quad + N_{rc}[\Theta^4(\eta,\xi;q) - \theta_a^4], \quad \text{at } \eta = 1.\end{aligned} \tag{18}$$

Then, using the above definitions, we construct the so-called zeroth-order deformation equation

$$\begin{aligned}&(1-q)\mathscr{L}[\Theta(\xi;\eta;q) - \theta_0(\eta,\xi)] \\ &\quad = q\hbar H(\eta,\xi)\mathscr{A}[\Theta(\eta,\xi;q)], \quad q \in [0,1],\end{aligned} \tag{19a}$$

subject to the initial condition

$$\Theta = 1, \quad \text{at } \xi = 0, \text{ for } \eta < 1 \text{ and } q \in [0,1] \tag{19b}$$

and boundary conditions

$$\Theta_\eta = 0, \quad \text{at } \eta = 0, \tag{19c}$$

$$(1-q)\mathscr{L}_b[\Theta - \theta_0] = q\hbar_b H_b(\xi)\mathscr{B}[\Theta(\eta,\xi;q)], \quad \text{at } \eta = 1, \tag{19d}$$

where $q \in [0,1]$ is an embedding parameter, $\hbar$ and $\hbar_b$ are auxiliary non-zero parameters, $H(\eta,\xi)$ and $H_b(\xi)$ are auxiliary functions, respectively. Obviously, when $q = 0$, the system of the above equations has the solution

$$\Theta(\eta,\xi;0) = \theta_0(\eta,\xi). \tag{20}$$

When $q = 1$, it is equivalent to the system of the original equations (11a)–(11d), provided

$$\Theta(\eta,\xi;1) = \theta(\eta,\xi). \tag{21}$$

Thus, as the embedding parameter q increases from 0 to 1, the mapping $\Theta(\eta,\xi;q)$ varies from the known initial approximation $\theta_0(\eta,\xi)$ to the unknown solution $\theta(\eta,\xi)$.

Expanding $\Theta(\eta,\xi;q)$ in Taylor series with respect to the embedding parameter q, we have

$$\Theta(\eta,\xi;q) = \Theta(\eta,\xi;0) + \sum_{m=1}^{+\infty}\theta_m(\eta,\xi)q^m, \tag{22}$$

where

$$\theta_m(\eta,\xi) = \frac{1}{m!}\frac{\partial^m\Theta(\eta,\xi;q)}{\partial q^m}\bigg|_{q=0}.$$

Obviously, it is important to ensure that the above series is convergent at $q = 1$. Fortunately, there exist the two auxiliary parameters $\hbar$ and $\hbar_b$, which provide us with a simple way to control and adjust the convergence of the series solution, as pointed out by Liao [15]. Assume that the auxiliary functions $H(\eta,\xi)$ and $H_b(\xi)$, and the two auxiliary parameters $\hbar$ and $\hbar_b$ are properly chosen so that the series (22) is convergent at $q = 1$. Then, using (20) and (21), we have the solution series

$$\theta(\eta,\xi) = \theta_0(\eta,\xi) + \sum_{m=1}^{+\infty}\theta_m(\eta,\xi). \tag{23}$$

Note that the term $\theta_m(\eta,\xi)$ in the above series is unknown at present. To deduce the governing equation for $\theta_m(\eta,\xi)$ and the corresponding initial and boundary conditions, we first differentiate the zeroth-order deformation equations (19a)–(19d) m times with respect to the embedding parameter q, then divide by $m!$, and finally set $q = 0$. In this way, we have the so-called high-order deformation equation

$$\mathscr{L}[\theta_m(\eta,\xi) - \chi_m\theta_{m-1}(\eta,\xi)] = \hbar H(\eta,\xi)R_m(\eta,\xi), \tag{24a}$$

subject to the initial condition

$$\theta_m(\eta,\xi) = 0, \quad \text{at } \xi = 0, \text{ for } \eta < 1 \tag{24b}$$

and the boundary conditions

$$\frac{\partial\theta_m}{\partial\eta} = 0, \quad \text{at } \eta = 0, \text{ for } 0 < \xi \leqslant 1, \tag{24c}$$

$$\mathscr{L}_b[\theta_m - \chi_m\theta_{m-1}] = \hbar_b H_b(\xi)G_m(\xi), \quad \text{at } \eta = 1, \text{ for } 0 < \xi \leqslant 1, \tag{24d}$$

where

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1 \end{cases} \tag{24e}$$

and

$$R_m(\eta,\xi) = \frac{\partial^2\theta_{m-1}}{\partial\eta^2} + \left(\frac{2}{\eta}\right)\frac{\partial\theta_{m-1}}{\partial\eta} - \alpha(1-\xi)\frac{\partial\theta_{m-1}}{\partial\xi} + \beta\sum_{n=0}^{m-1}\left[\theta_n\frac{\partial^2\theta_{m-1-n}}{\partial\eta^2} + \left(\frac{2}{\eta}\right)\theta_n\frac{\partial\theta_{m-1-n}}{\partial\eta} + \frac{\partial\theta_n}{\partial\eta}\frac{\partial\theta_{m-1-n}}{\partial\eta}\right], \tag{24f}$$

$$G_m(\xi) = \frac{\partial\theta_{m-1}}{\partial\eta} + Bi\theta_{m-1} - (1-\chi_m)(Bi\theta_a + N_{rc}\theta_a^4) + \sum_{n=0}^{m-1}\left[\beta\theta_n\frac{\partial\theta_{m-1-n}}{\partial\eta} + N_{rc}\Delta_n(\xi)\Delta_{m-1-n}(\xi)\right], \quad \text{at } \eta = 1, \tag{24g}$$

in which

$$\Delta_n(\xi) = \sum_{j=0}^{n}\theta_j(1,\xi)\theta_{n-j}(1,\xi).$$

Note that, substituting (22) into the zero-order deformation equations (19a)–(19d), and equating the coefficients of the like powers of q, we can get exactly the same equations as above.

To satisfy the initial condition (24b), we choose the following auxiliary functions:

$$H(\eta,\xi) = H_b(\xi) = \xi. \tag{25}$$

Then, it is easy to solve the system of the above linear high-order deformation equations. Let $\theta_m^*(\eta,\xi)$ denote a special solution of Eqs. (24a) and (24b). The corresponding general solution reads

$$\theta_m(\eta,\xi) = \theta_m^*(\eta,\xi) + C_1 + \frac{C_2}{\eta}, \tag{26}$$

where C_1 and C_2 are integral constants. To satisfy the boundary condition (24c), it holds

$$C_2 = 0.$$

The integral constant C_1 is determined by the boundary condition (24d).

It is found that θ_m can be expressed by

$$\theta_m(\eta,\xi) = \sum_{n=0}^{m+1} b_{m,n}(\xi)\eta^{2n}, \tag{27}$$

where $b_{m,n}(\xi)$ are dependent upon ξ. Substituting it into the high-order deformation equations (24a)–(24d) and equating the coefficients of the same power of η, we deduce the following recurrence formulas:

$$b_{m,0}(\xi) = \sigma_m(\xi) + \chi_m b_{m-1,0}(\xi), \tag{28a}$$

$$b_{m,m+1}(\xi) = \frac{\hbar\xi S_{m,m}(\xi)}{(2m+2)(2m+3)}, \tag{28b}$$

$$b_{m,n}(\xi) = \frac{\hbar\xi S_{m,n-1}(\xi)}{2n(2n+1)} + \chi_m b_{m-1,n}(\xi), \quad 1 \leqslant n \leqslant m, \tag{28c}$$

where

$$S_{m,n}(\xi) = (2n+2)(2n+3)\chi_{m+1-n} b_{m-1,n+1}(\xi) - \alpha(1-\xi) b'_{m-1,n}(\xi) + \beta[A_{m,n}(\xi) + \chi_{n+1} B_{m,n}(\xi)], \tag{28d}$$

$$A_{m,n}(\xi) = \sum_{n=0}^{m-1} \sum_{j=\max\{1,k-n\}}^{\min\{m-n,k+1\}} 2j(2j+1) b_{n,k+1-j}(\xi) b_{m-1-n,j}(\xi), \tag{28e}$$

$$B_{m,n}(\xi) = \sum_{n=0}^{m-1} \sum_{j=\max\{1,k-n\}}^{\min\{m-n,k\}} 4j(k+1-j) b_{n,k+1-j}(\xi) b_{m-1-n,j}(\xi), \tag{28f}$$

for $0 \leqslant n \leqslant m$, and

$$\sigma_m(\xi) = \left(\frac{\xi}{Bi}\right)\left[\hbar_b G_m(\xi) - \hbar \sum_{n=1}^{m+1} \left(1 + \frac{Bi}{2n}\right) \frac{S_{m,n-1}(\xi)}{(2n+1)}\right], \tag{28g}$$

$$G_m(\xi) = \mu_{m-1}(\xi) + Bi\delta_{m-1}(\xi) - (1-\chi_m)(Bi\theta_a + N_{rc}\theta_a^4) + \sum_{n=0}^{m-1} [\beta\delta_n(\xi)\mu_{m-1-n}(\xi) + N_{rc}\Delta_n(\xi)\Delta_{m-1-n}(\xi)], \tag{28h}$$

$$\delta_m(\xi) = \sum_{n=0}^{m+1} b_{m,n}(\xi), \tag{28i}$$

$$\mu_m(\xi) = \sum_{n=1}^{m+1} (2n) b_{m,n}(\xi), \tag{28j}$$

$$\Delta_m(\xi) = \sum_{n=0}^{m} \delta_n(\xi)\delta_{m-n}(\xi). \tag{28k}$$

The first two coefficients are given by the initial approximation (13), i.e.

$$b_{0,0}(\xi) = 1 + (\theta_a - 1)\xi, \quad b_{0,1}(\xi) = \gamma\xi(1-\xi). \tag{29}$$

Thus, using the above recurrence formulas and the first two coefficients (29), we can obtain all coefficients $b_{m,n}(\xi)$ of the series solution, one by one. So, the above recurrence formulas in fact provide us an explicit expression of the series solution of the nonlinear model of combined convective and radiative cooling of a spherical body.

Table 1
Values of $\hbar$, $\hbar_b$, α and γ for different cases

β	N_{rc}	Bi	θ_a	α	γ	$\hbar$	$\hbar_b$
0	0	1	0	1	−4/5	−1/3	−1/3
1	0	1/2	0	17/20	−2/5	−1/3	−1/3
1	0	1	0	13/10	−7/10	−1/3	−1/3
1	0	2	0	2	−6/5	−1/3	−1/3
1/2	0	1	0	6/5	−4/5	−1/3	−1/3
2	0	1	0	4/5	−1/2	−1/3	−1/3
1	1/4	1/2	1/2	6/5	−1/4	−1/5	−1/5
1	1/2	1/2	1/2	13/10	−3/10	−1/5	−1/5

4. Result analysis

Note that, right now, there are still four parameters α, γ, $\hbar$ and $\hbar_b$ that are unknown. The parameter γ is determined by the minimum value of the square residual error of the guess approximation $\theta_0(\eta,\xi)$ for the boundary condition at $\eta = 1$, integrated over the whole region $\xi \in [0,1]$, i.e.

$$\frac{\partial}{\partial\gamma} \int_0^1 \left[(1+\beta\theta_0)\frac{\partial\theta_0}{\partial\eta} + Bi(\theta_0 - \theta_a) + N_{rc}(\theta_0^4 - \theta_a^4)\right]^2 d\xi = 0.$$

Then, the value of α is determined by the minimum of the square residual error of the guess approximation $\theta_0(\eta,\xi)$ for the governing equation, integrated over the whole region $\xi \in [0,1]$ and $x \in [0,1]$, i.e.

$$\frac{\partial}{\partial\alpha} \int_0^1 \int_0^1 \left\{\alpha(1-\xi)\theta_\xi - \frac{1}{\eta^2}\frac{\partial}{\partial\eta}\left[(1+\beta\theta_0)\eta^2 \frac{\partial\theta_0}{\partial\eta}\right]\right\}^2 d\xi\, d\eta = 0.$$

For example, we have approximately $\gamma = -7/10$, $\alpha = 13/10$ when $\beta = 1$, $Bi = 1$, $N_{rc} = 0$ and $\theta_a = 0$. The values of α and γ for different cases are listed in Table 1.

It is important to ensure that the solution series (23) converges. Fortunately, we have freedom to choose the values of the auxiliary parameters $\hbar$ and $\hbar_b$. These two parameters provide us with a simple way to adjust and control the convergence region and rate of the solution series, as mentioned by Liao [15]. Let $\hbar$ and $\hbar_b$ be unknown variable. Then, the square residual errors of the high-order approximation for the governing equation and the boundary condition are functions of $\hbar$ and $\hbar_b$. By plotting the curves of square residual errors versus $\hbar$ and $\hbar_b$, it is straightforward to choose proper values of $\hbar$ and $\hbar_b$ so as to ensure that the solution series converges, as suggested by Liao [15]. It is found that $\hbar$ and $\hbar_b$ must be negative. Besides, the solution series diverges if the absolute values of $\hbar$ and $\hbar_b$ are too large. On the other hand, the solution series converges rather slowly, if the absolute values of $\hbar$ and $\hbar_b$ are too small. When the solution series converges slowly or diverges, the so-called homotopy-Padé technique [15] can be employed to accelerate the convergence or to gain a convergent solution.

When $\beta = 0$ and $N_{rc} = 0$, our HAM solution agrees well with the exact solution (8), as shown in Figs. 1 and 2 for the spatial and temporal distributions of temperature on the surface of the body, respectively. Note that, the 30th-order HAM approximation agrees with the exact solution (8) for dimensionless time $0.35 \leqslant \tau < +\infty$. For small τ, the series solution converges slowly, and thus the homotopy-Padé technique is applied to accelerate the convergence rate. Note that the [15,15] homotopy-Padé approximation agrees well with the exact solution (8). This verifies the validity of the homotopy analysis method for the unsteady

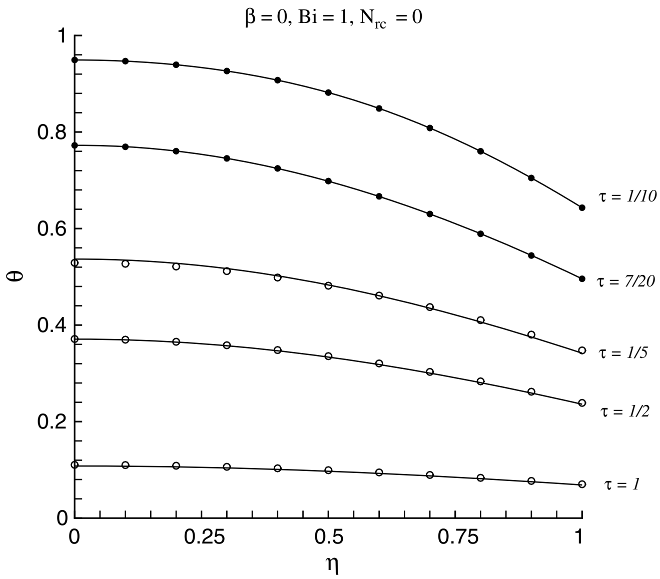

Fig. 1. Comparison of the exact solution (8) with the analytic approximation when $Bi = 1$, $\beta = 0$, $N_{rc} = 0$, $\theta_a = 0$ by means of $\gamma = -4/5$, $\alpha = 1$, $\hbar = \hbar_b = -1/3$ at the dimensionless time $\tau = 1/10$, 1/5, 7/20, 1/2, 1. Solid line: exact solution; open circle: 30th-order analytic approximation; filled circle: [15, 15] homotopy-Padé approximation of analytic results.

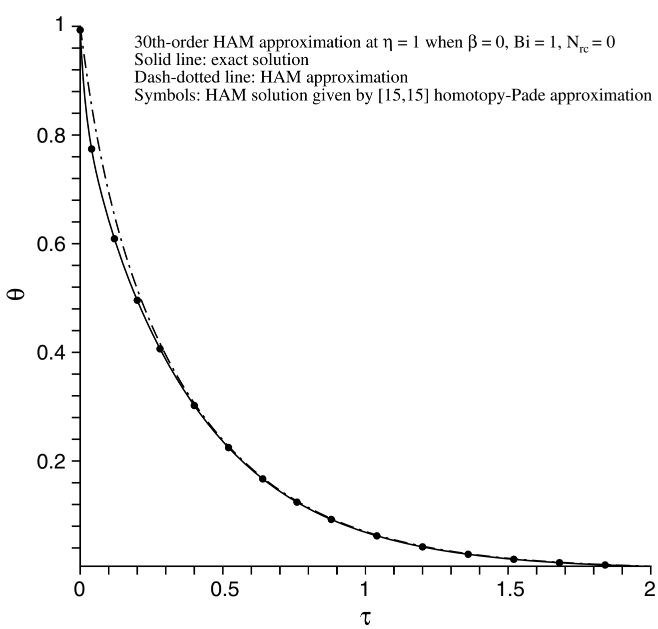

Fig. 2. The 30th-order HAM approximations of θ at the boundary $\eta = 1$ when $Bi = 1$, $\beta = 0$, $N_{rc} = 0$, $\theta_a = 0$ by means of $\gamma = -4/5$, $\alpha = 1$, $\hbar = \hbar_b = -1/3$. Solid line: exact solution (8); dash-dotted line: 30th-order HAM approximation; symbols: [15, 15] homotopy-Padé approximation.

heat transfer problem under consideration. Different from traditional methods for linear problems, our analytic approach does *not* use the concept of eigenvalues and eigenfunctions at all, and besides is valid for nonlinear problems, as shown below.

The spatial distributions of temperature in case $\beta = 1$ and $Bi = 0.5$, 1 and 2 are as shown in Figs. 3–5, respectively. Similarly, our 30th-order approximation agrees well with the numerical solutions when $0.35 \leqslant \tau < +\infty$, and the homotopy-Padé technique is applied to accelerate the convergence rate for small time. The temporal variations of the temperature on the surface of the body are as shown in Fig. 6, which indicates that the temperature on the surface

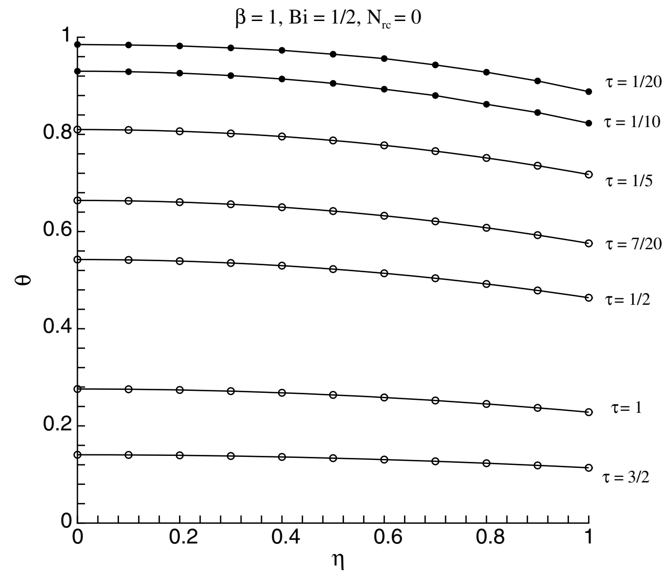

Fig. 3. Comparison of the numerical results with analytic approximations when $Bi = 1/2$, $\beta = 1$, $N_{rc} = 0$, $\theta_a = 0$ by means of $\gamma = -2/5$, $\alpha = 17/20$, $\hbar = \hbar_b = -1/3$ at the dimensionless time $\tau = 1/20$, 1/10, 1/5, 7/20, 1/2, 1, 3/2. Solid line: numerical results; open circle: 30th-order HAM approximation; filled circle: [15, 15] homotopy-Padé approximation of series solution.

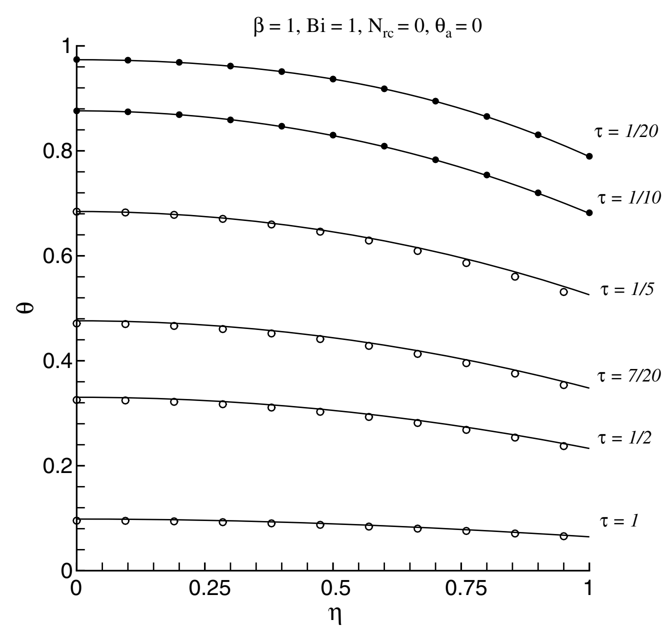

Fig. 4. Comparison of the numerical results with analytic approximations when $Bi = 1$, $\beta = 1$, $N_{rc} = 0$, $\theta_a = 0$ by means of $\gamma = -7/10$, $\alpha = 13/10$, $\hbar = \hbar_b = -1/3$ at the dimensionless time $\tau = 1/20$, 1/10, 1/5, 7/20, 1/2, 1. Solid line: numerical results; open circle: 30th-order analytic approximation; filled circle: [15, 15] homotopy-Padé approximation of series solution.

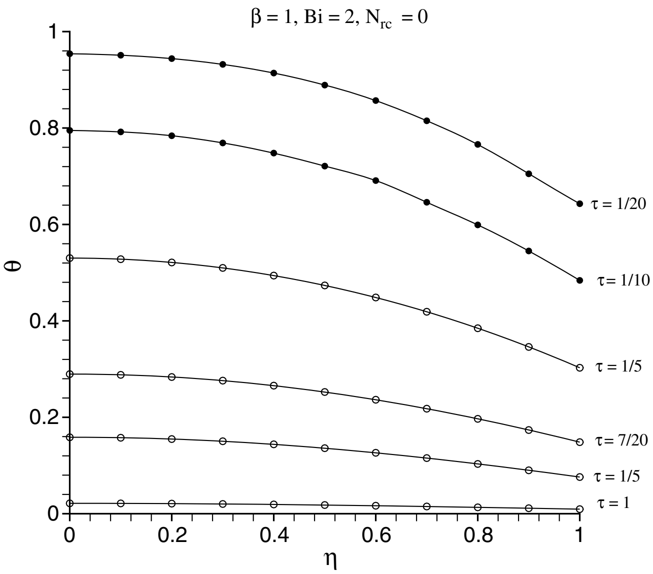

Fig. 5. Comparison of the numerical results with analytic approximation when $Bi = 2$, $\beta = 1$, $N_{rc} = 0$, $\theta_a = 0$ by means of $\gamma = -6/5$, $\alpha = 2$, $\hbar = \hbar_b = -1/3$ at the dimensionless time $\tau = 1/20$, 1/10, 1/5, 7/20, 1/2, 1. Solid line: numerical results; open circle: 30th-order analytic approximation; filled circle: [15, 15] homotopy-Padé approximation of series solution.

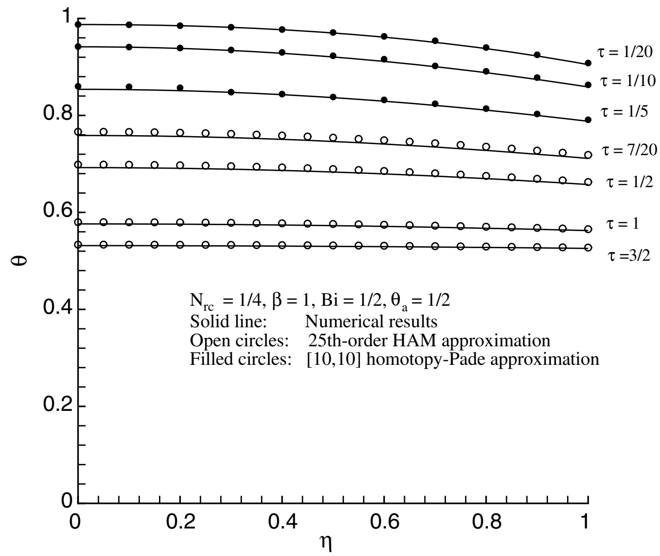

Fig. 7. Comparison of the numerical results with analytic approximation when $Bi = 1/2$, $\beta = 1$, $N_{rc} = 1/4$, $\theta_a = 1/2$ by means of $\gamma = -1/4$, $\alpha = 6/5$, $\hbar = -1/5$, $\hbar_b = -1/5$ at the dimensionless time $\tau = 1/20$, 1/10, 1/5, 7/20, 1/2, 1. Solid line: numerical results; open circle: 25th-order analytic approximation; filled circle: [10, 10] homotopy-Padé approximation of series solution.

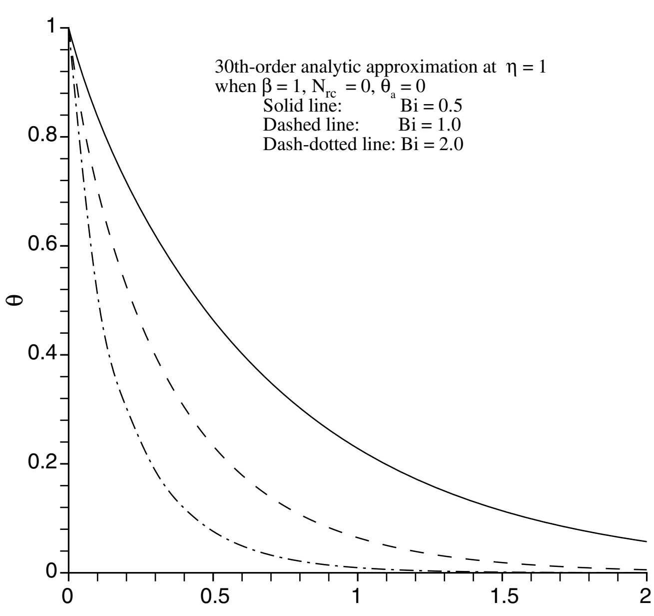

Fig. 6. The 30th-order HAM approximations of θ at the boundary $\eta = 1$ for different values of Bi when $\beta = 1$, $N_{rc} = 0$, $\theta_a = 0$ by means of $\hbar = \hbar_b = -1/3$. Solid line: $Bi = 0.5$; dashed line: $Bi = 1$; dash-dotted line: $Bi = 2$.

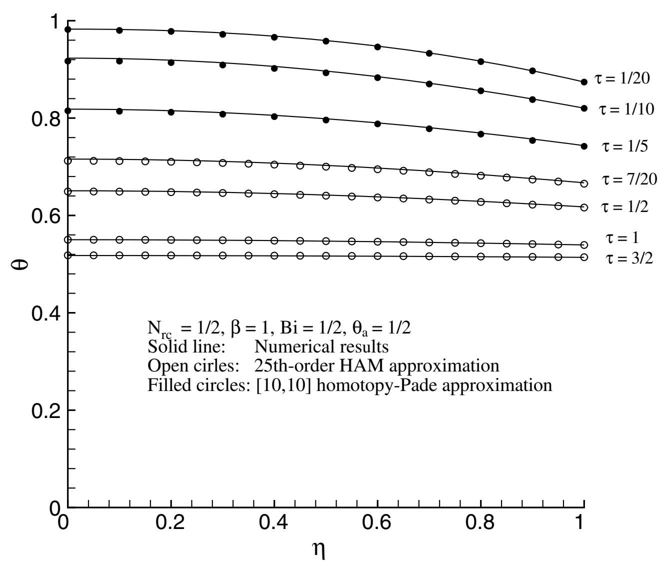

Fig. 8. Comparison of the numerical results with analytic approximation when $Bi = 1/2$, $\beta = 1$, $N_{rc} = 1/2$, $\theta_a = 1/2$ by means of $\gamma = -3/10$, $\alpha = 13/10$, $\hbar = -1/5$, $\hbar_b = -1/5$ at the dimensionless time $\tau = 1/20$, 1/10, 1/5, 7/20, 1/2, 1. Solid line: numerical results; open circle: 25th-order analytic approximation; filled circle: [10,10] homotopy-Padé approximation of series solution.

decays more quickly for large values of the Biot number Bi. For the cases of combined convective and radiative cooling, for instance $N_{rc} = 1/4$ or $N_{rc} = 1/2$ with $\beta = 1$, $Bi = 1/2$, $\theta_a = 1/2$, our series solutions also give accurate results, as shown in Figs. 7 and 8 for the spatial variation of the temperature. The corresponding temporal variations on the surface of the body are as shown in Fig. 9, which indicates that the temperature on the surface decays more quickly for larger value of the radiation–conduction parameter N_{rc}. All of these results verify the validity of the homotopy analysis method for the unsteady nonlinear heat transfer problems.

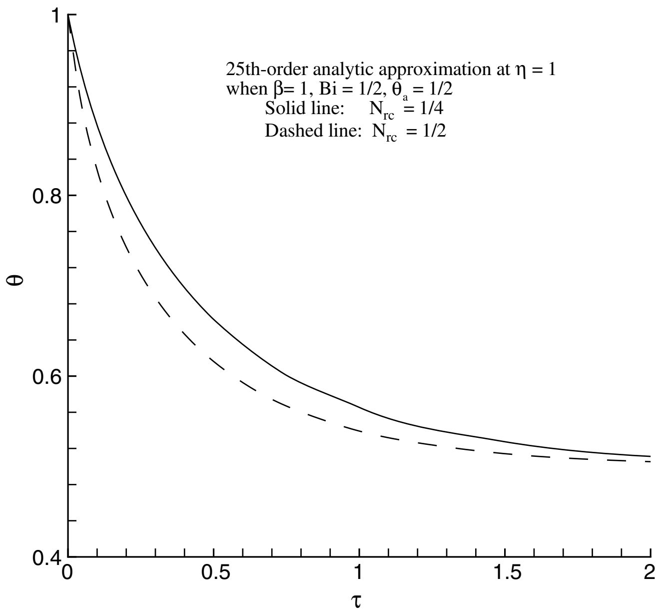

Fig. 9. The 25th-order HAM approximations of θ at the boundary $\eta = 1$ for different values of N_{rc} when $\beta = 1$, $Bi = 1/2$, $\theta_a = 1/2$ by means of $\hbar = \hbar_b = -1/5$. Solid line: $N_{rc} = 0.25$; dashed line: $N_{rc} = 0.5$.

5. Conclusions and discussions

A new analytic approach based on the homotopy analysis method is proposed to solve a nonlinear model of combined convective and radiative cooling of a spherical body. An explicit series solution is given, which agrees well with the exact (when possible) or numerical solutions. Different from traditional analytic techniques, our approach is independent of the concept of eigenfunctions and eigenvalues, and besides is valid for nonlinear problems in general. Note that the initial condition is satisfied simply by means of the so-called auxiliary function. This analytic approach provides us with a new way to obtain explicit series solutions of unsteady nonlinear heat transfer problems, which are valid for all dimensionless times $0 \leqslant \tau < +\infty$.

Our series solution gives accurate spatial and temporal variations of the temperature. They show that, for the nonlinear model of combined convective and radiative cooling of a spherical body, the temperature on the surface of the body decays more quickly for large values of the Biot number Bi and/or the radiation–conduction parameter N_{rc}.

Note that it is always fine to give an explicit analytic solution of a nonlinear problem, even if one can apply numerical techniques to solve it. Besides, nonlinear problems are rather complicated, and it is a great challenge to solve some of them even by means of numerical methods, as shown by Liao [39]. Thus, it is valuable to develop new, more powerful analytic tools for strongly nonlinear problems. The current work might provide a new approach for solving nonlinear partial differential equations in general.

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